Kinetic Equations

Solution to the Exercises

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Exercise 1

We recall that the collision term of the (general) Boltzmann equation for hard sphere interactions is:

$$Q\left(f,f\right)(v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2}\left(f\left(v'\right)f\left(v'_*\right) - f\left(v\right)f\left(v_*\right)\right)\left|(v-v_*)\cdot\omega\right| d\omega\ dv_*. \tag{1}$$

We consider now an homogeneous solution f of the Boltzmann equation (which does not depend on the position variable x) and radial in velocity (which depends only on the norm $|v|$ of the velocity variable v).

- Under those hypotheses, show that the collision term (1) of the Boltzmann equation writes:

  $$\begin{aligned} &Q\left(f,f\right)(v) = \\ &= 4\pi^2\int_0^{+\infty}\int_0^{\pi}\int_0^{\pi}\left(f\left(\sqrt{r^2\sin^2\theta + r_1^2\cos^2\theta_1}\right)f\left(\sqrt{r_1^2\sin^2\theta_1 + r^2\cos^2\theta}\right) - \right.\\ &\qquad\left. - f\left(r\right)f\left(r_1\right)\right)\left|r_1\cos\theta_1 - r\cos\theta\right|\sin\theta\sin\theta_1 r_1^2 d\theta\ d\theta_1\ dr_1, \end{aligned} \tag{2}$$

  where r denotes $|v|$.

  Hint: Denote as r_1 the norm of the velocity v_*, θ the angle between the velocity v and ω, and θ_1 the angle between the velocity v_* and ω.

- Considering the transformation $\varphi : (r, r_1, \theta, \theta_1) \mapsto (r', r'_1, \theta', \theta'_1)$ defined through the system:

  $$\begin{cases} r'\cos\theta' = r_1\cos\theta_1, \\ r'\sin\theta' = r\sin\theta, \\ r'_1\cos\theta'_1 = r\cos\theta, \\ r'_1\sin\theta'_1 = r_1\sin\theta_1, \end{cases} \tag{3}$$

  show that the collision term (2) can be abbreviated as:

  $$C\int_0^{+\infty}\int_0^{\pi}\int_0^{\pi}\Big(f(t,r')f(t,r'_1) - f(t,r)f(t,r_1)\Big)V(r,r_1,\theta,\theta_1)r_1^2 d\theta d\theta_1 dr_1, \tag{4}$$

  with $V(r,r_1,\theta,\theta_1) = \big|r_1\cos\theta_1 - r\cos\theta\big|\sin\theta\sin\theta_1$.

Proof. We first of all perform a rotation. Indeed, for any vector $v \in \mathbb{R}^3$ with $|v| = r$ we get that there exists a rotation of the space R such that $v = rRe_3$, where $e_3 = (0,0,1)$. We can then write

$$(v - v_*)\cdot\omega = (rRe_3 - v_*)\cdot\omega = \big(re_3 - R^{-1}v_*\big)\cdot R^{-1}\omega. \tag{5}$$

Therefore, applying a suitable change of variables and using that f is radial and therefore invariant under rotation we get[1]

$$Q\left(f,f\right)\left(v\right)=\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}\left(f\left(v'\right)f\left(v'_*\right)-f\left(v\right)f\left(v_*\right)\right)\left|\left(re_3-v_*\right)\cdot\omega\right|d\omega\ dv_*. \tag{6}$$

Now recall the following definitions:

$$e_r\left(\varphi,\theta\right):=\begin{pmatrix}\sin\theta\cos\varphi\\ \sin\theta\sin\varphi\\ \cos\theta\end{pmatrix}, \tag{7}$$

$$e_\theta\left(\varphi,\theta\right):=\partial_\theta\ e_r\left(\varphi,\theta\right)=\begin{pmatrix}\cos\theta\cos\varphi\\ \cos\theta\sin\varphi\\ -\sin\theta\end{pmatrix}, \tag{8}$$

$$e_\varphi\left(\varphi\right):=\frac{1}{\cos\theta}\partial_\varphi e_r\left(\varphi,\theta\right)=\begin{pmatrix}-\sin\varphi\\ \cos\varphi\\ 0\end{pmatrix}. \tag{9}$$

We now want to rewrite Q in spherical coordinates; to do so, we first write $\omega=e_r\left(\varphi,\theta\right)$. Q becomes then of the form

$$Q\left(f,f\right)\left(v\right)=\int_{\mathbb{R}^3}\int_{-\pi}^{\pi}\int_0^{\pi}\left(f\left(v'\right)f\left(v'_*\right)-f\left(v\right)f\left(v_*\right)\right)\cdot \tag{10}$$

$$\cdot\left|\left(re_3-v_*\right)\cdot e_r\right|\sin\theta d\theta\ d\varphi\ dv_*. \tag{11}$$

We then turn our attention at the integral in v_*; we write v_* in spherical coordinates with respect to e_r, e_θ and e_φ so that θ_1 will be the angle between v_* and ω. We will therefore use the change of variables for v_* defined as

$$v_*=r_1\left(\cos\theta_1 e_r\left(\varphi,\theta\right)+\sin\theta_1\cos\varphi_1 e_\theta\left(\varphi,\theta\right)+\sin\theta_1\sin\varphi_1 e_\varphi\left(\varphi,\theta\right)\right). \tag{12}$$

Given that e_r, e_θ and e_φ represent an orthonormal system, the change of variable is given as $dv_*=r_1^2\sin\theta_1 d\theta_1\ d\varphi_1\ dr_1$. Furthermore, we have some interesting properties of this change of variables. First of all we can rewrite $\left(re_3-v_*\right)\cdot e_r=r\cos\theta-r_1\cos\theta_1$. We then want to understand under these new coordinates, how one can write v' and v'_*. Recall that by definition

$$\begin{cases}v'=re_3-\left(re_3-v_*\right)\cdot\omega\ \omega,\\ v'_*=v_*+\left(re_3-v_*\right)\cdot\omega\ \omega.\end{cases} \tag{13}$$

[1]Notice that here we also used the fact that given that we are applying the same transformation to v_* and ω, the change of variable acts on v' and v'_* mapping those to $R^{-1}v'$ and $R^{-1}v'_*$ respectively.

This implies in particular

$$\left|v'\right|^2 = \left|re_3 - (re_3 - v_*)\cdot\omega\ \omega\right|^2 = r^2 - 2re_3\cdot\omega\,(re_3 - v_*)\cdot\omega + \left|(re_3 - v_*)\cdot\omega\right|^2 \tag{14}$$
$$= r^2 - 2r\cos\theta\,(r\cos\theta - r_1\cos\theta_1) + (r\cos\theta - r_1\cos\theta_1)^2 \tag{15}$$
$$= r^2 - (r\cos\theta + r_1\cos\theta_1)\,(r\cos\theta - r_1\cos\theta_1) \tag{16}$$
$$= r^2 - r^2\cos^2\theta + r_1^2\cos^2\theta_1 \tag{17}$$
$$= r^2\sin^2\theta + r_1^2\cos^2\theta_1, \tag{18}$$
$$\left|v_*'\right|^2 = |v|^2 + |v_*|^2 - \left|v'\right|^2 = r^2 + r_1^2 - \left(r^2\sin^2\theta + r_1^2\cos^2\theta_1\right) \tag{19}$$
$$= r^2\cos^2\theta + r_1^2\sin^2\theta_1. \tag{20}$$

With all these informations, we are finally able to rewrite Q as

$$Q\,(f,f)\,(v) = \tag{21}$$
$$= \int_0^{+\infty}\int_{-\pi}^{\pi}\int_0^{\pi}\int_{-\pi}^{\pi}\int_0^{\pi}\left(f\left(\sqrt{r^2\sin^2\theta + r_1^2\cos^2\theta_1}\right)f\left(\sqrt{r^2\cos^2\theta + r_1^2\sin^2\theta_1}\right)\right. \tag{22}$$
$$\left.-f\,(r)\,f\,(r_1)\right)|r\cos\theta - r_1\cos\theta_1|\sin\theta\sin\theta_1 r_1^2 d\theta\ d\varphi\ d\theta_1\ d\varphi_1\ dr_1 \tag{23}$$
$$= 4\pi^2\int_0^{+\infty}\int_0^{\pi}\int_0^{\pi}\left(f\left(\sqrt{r^2\sin^2\theta + r_1^2\cos^2\theta_1}\right)f\left(\sqrt{r^2\cos^2\theta + r_1^2\sin^2\theta_1}\right)\right. \tag{24}$$
$$\left.-f\,(r)\,f\,(r_1)\right)|r\cos\theta - r_1\cos\theta_1|\sin\theta\sin\theta_1 r_1^2 d\theta\ d\theta_1\ dr_1. \tag{25}$$

For the next step consider the transformation induced by

$$\begin{cases} r'\cos\theta' = r_1\cos\theta_1, \\ r'\sin\theta' = r\sin\theta, \\ r_1'\cos\theta_1' = r\cos\theta, \\ r_1'\sin\theta_1' = r_1\sin\theta_1. \end{cases} \tag{26}$$

In this case we clearly have that

$$\sqrt{r^2\sin^2\theta + r_1^2\cos^2\theta_1} = \sqrt{(r')\sin^2\theta' + (r')\cos^2\theta'} = r', \tag{27}$$
$$\sqrt{r^2\cos^2\theta + r_1^2\sin^2\theta_1} = \sqrt{(r_1')\sin^2\theta_1' + (r_1')\cos^2\theta_1'} = r_1', \tag{28}$$

which gives us (4).

□

Exercise 2

We consider now the gain term of the collision operator, that is the part:

$$\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} f' f_*' B(v - v_*, \omega) d\omega dv_*$$

in the right-hand side of the Boltzmann equation. In the case of hard sphere interactions with a solution which is homogeneous and radial in velocity, we saw that this term can be written as:

$$J(f) = \int_0^{+\infty}\int_0^{\pi}\int_0^{\pi} f\Big(t, \sqrt{r^2 \sin^2\theta + r_1^2\cos^2\theta_1}\Big) f\Big(t, \sqrt{r_1^2\sin^2\theta_1 + r^2\cos^2\theta}\Big) \times \big|r_1\cos\theta_1 - r\cos\theta\big| \sin\theta \sin\theta_1 r_1^2 d\theta d\theta_1 dr_1. \tag{29}$$

- Considering $x = \cos\theta$ and $y = \cos\theta_1$, show that (29) is equal to

$$2\int_0^{+\infty}\int_0^1\int_0^1 f\Big(t, \sqrt{r^2 - r^2x^2 + r_1^2y^2}\Big) f\Big(t, \sqrt{r_1^2 - r_1^2y^2 + r^2x^2}\Big) \times \big(|r_1y - rx| + |r_1y + rx|\big) r_1^2 dy dx dr_1. \tag{30}$$

- Considering $u = \sqrt{r^2 - r^2x^2 + r_1^2y^2}$ and $v = \sqrt{r_1^2 - r_1^2y^2 + r^2x^2}$, show that (29) is equal to

$$4\int_0^{+\infty}\int_0^{+\infty} f(t,u)f(t,v)G(r,u,v)uvdudv, \tag{31}$$

where G is defined as:

$$\begin{cases} G(r,u,v) & = 0 & \text{if } u^2+v^2 \leqslant r^2, \\ G(r,u,v) & = 1 & \text{if } u \geqslant r,\ v \geqslant r, \\ G(r,u,v) & = v/r & \text{if } u \geqslant r,\ v \leqslant r, \\ G(r,u,v) & = u/r & \text{if } u \leqslant r,\ v \geqslant r, \\ G(r,u,v) & = \sqrt{u^2+v^2-r^2}/r & \text{if } u^2+v^2 \geqslant r^2,\ u \leqslant r,\ v \leqslant r. \end{cases}$$

Proof. Consider J as defined in (29). Consider the change of variables give by

$$\begin{cases} x = \cos\theta, \\ y = \cos\theta_1. \end{cases} \tag{32}$$

In order to apply this change of variables we split the integral in J to get

$$
\begin{aligned}
J(f)(r) = & \int_0^{+\infty}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \left(f\left(\sqrt{r^2\sin^2\theta + r_1^2\cos^2\theta_1}\right) f\left(\sqrt{r^2\cos^2\theta + r_1^2\sin^2\theta_1}\right) \right. && (33)\\
& \left. - f(r) f(r_1)\right) |r\cos\theta - r_1\cos\theta_1| \sin\theta\sin\theta_1 r_1^2 d\theta\ d\theta_1\ dr_1 && (34)\\
& + \int_0^{+\infty}\int_0^{\frac{\pi}{2}}\int_{\frac{\pi}{2}}^{\pi} \left(f\left(\sqrt{r^2\sin^2\theta + r_1^2\cos^2\theta_1}\right) f\left(\sqrt{r^2\cos^2\theta + r_1^2\sin^2\theta_1}\right) \right. && (35)\\
& \left. - f(r) f(r_1)\right) |r\cos\theta - r_1\cos\theta_1| \sin\theta\sin\theta_1 r_1^2 d\theta\ d\theta_1\ dr_1 && (36)\\
& + \int_0^{+\infty}\int_{\frac{\pi}{2}}^{\pi}\int_0^{\frac{\pi}{2}} \left(f\left(\sqrt{r^2\sin^2\theta + r_1^2\cos^2\theta_1}\right) f\left(\sqrt{r^2\cos^2\theta + r_1^2\sin^2\theta_1}\right) \right. && (37)\\
& \left. - f(r) f(r_1)\right) |r\cos\theta - r_1\cos\theta_1| \sin\theta\sin\theta_1 r_1^2 d\theta\ d\theta_1\ dr_1 && (38)\\
& + \int_0^{+\infty}\int_{\frac{\pi}{2}}^{\pi}\int_{\frac{\pi}{2}}^{\pi} \left(f\left(\sqrt{r^2\sin^2\theta + r_1^2\cos^2\theta_1}\right) f\left(\sqrt{r^2\cos^2\theta + r_1^2\sin^2\theta_1}\right) \right. && (39)\\
& \left. - f(r) f(r_1)\right) |r\cos\theta - r_1\cos\theta_1| \sin\theta\sin\theta_1 r_1^2 d\theta\ d\theta_1\ dr_1 && (40)\\
& = \int_0^{+\infty}\int_0^{1}\int_0^{1} \left(f\left(\sqrt{r^2(1+x^2) + r_1^2 y^2}\right) f\left(\sqrt{r^2x^2 + r_1^2(1-y^2)}\right) \right. && (41)\\
& \left. - f(r) f(r_1)\right) |rx - r_1 y| r_1^2 dx\ dy\ dr_1 && (42)\\
& + \int_0^{+\infty}\int_0^{1}\int_{-1}^{0} \left(f\left(\sqrt{r^2(1+x^2) + r_1^2 y^2}\right) f\left(\sqrt{r^2x^2 + r_1^2(1-y^2)}\right) \right. && (43)\\
& \left. - f(r) f(r_1)\right) |rx - r_1 y| r_1^2 dx\ dy\ dr_1 && (44)\\
& + \int_0^{+\infty}\int_{-1}^{0}\int_0^{1} \left(f\left(\sqrt{r^2(1+x^2) + r_1^2 y^2}\right) f\left(\sqrt{r^2x^2 + r_1^2(1-y^2)}\right) \right. && (45)\\
& \left. - f(r) f(r_1)\right) |rx - r_1 y| r_1^2 dx\ dy\ dr_1 && (46)\\
& + \int_0^{+\infty}\int_{-1}^{0}\int_{-1}^{0} \left(f\left(\sqrt{r^2(1+x^2) + r_1^2 y^2}\right) f\left(\sqrt{r^2x^2 + r_1^2(1-y^2)}\right) \right. && (47)\\
& \left. - f(r) f(r_1)\right) |rx - r_1 y| r_1^2 dx\ dy\ dr_1 && (48)\\
& = 2\int_0^{+\infty}\int_0^{1}\int_0^{1} \left(f\left(\sqrt{r^2(1+x^2) + r_1^2 y^2}\right) f\left(\sqrt{r^2x^2 + r_1^2(1-y^2)}\right) \right. && (49)\\
& \left. - f(r) f(r_1)\right) \left(|rx - r_1 y| + |rx + r_1 y|\right) r_1^2 dx\ dy\ dr_1. && (50)
\end{aligned}
$$

We now want to apply the change of variables given by

$$
\begin{cases} \alpha = rx + r_1 y, \\ \beta = rx - r_1 y. \end{cases} \tag{51}
$$

The differential is given by $dxdy = \frac{2rr_1}{d}\alpha d\beta$, and therefore J becomes

$$
J(f)(r) = \int_0^{+\infty} \iint\limits_{\substack{0\leqslant\alpha+\beta\leqslant 2r \\ 0\leqslant\alpha-\beta\leqslant 2r_1}} f\left(\sqrt{r^2 - \alpha\beta}\right) f\left(\sqrt{r_1^2 + \alpha\beta}\right) \left(|\alpha| + |\beta|\right) \frac{r_1}{r} d\alpha\ d\beta\ dr_1. \tag{52}
$$

Finally we perform the change

$$\begin{cases} \alpha = \alpha, \\ u = \sqrt{r^2 - \alpha\beta}, \\ v = \sqrt{r_1^2 + \alpha\beta}, \end{cases} \tag{53}$$

with the differential give by $d\alpha d\beta dr_1 = \frac{2uv}{\alpha r_1} d\alpha du dv$. We then get

$$J(f)(r) = \iiint\limits_{\mathcal{D}} f(u) f(v) \left(|\alpha| + \left| \frac{r^2 - u^2}{\alpha} \right| \right) \frac{r_1}{r} \frac{2uv}{\alpha r_1} d\alpha \, du \, dv \tag{54}$$

$$= \frac{2}{r} \iiint\limits_{\mathcal{D}} f(u) f(v) \frac{\alpha^2 + |r^2 - u^2|}{\alpha^2} uv d\alpha \, du \, dv, \tag{55}$$

with $\mathcal{D}$ defined as

$$\mathcal{D} = \Big\{ (\alpha, u, v) \in (0, +\infty) \times (0, +\infty) \times (0, +\infty) \mid u^2 + v^2 \geqslant r^2, \tag{56}$$

$$\sqrt{|r^2 - u^2|} \leqslant \alpha \leqslant \min \left\{ r + u, \sqrt{u^2 + v^2 - r^2} + v \right\} \Big\}. \tag{57}$$

Given that

$$\int_A^B \frac{\alpha^2 + A^2}{\alpha^2} d\alpha = \frac{B^2 - A^2}{B}, \tag{58}$$

we get now

$$J(f)(r) = \frac{2}{r} \iint\limits_{u^2 + v^2 \geqslant r^2} \int_{\sqrt{|r^2 - u^2|}}^{\min\{r+u, \sqrt{u^2+v^2-r^2}+v\}} f(u) f(v) \frac{\alpha^2 + |r^2 - u^2|}{\alpha^2} uv d\alpha \, du \, dv, \tag{59}$$

$$= \frac{2}{r} \iint\limits_{u^2 + v^2 \geqslant r^2} \frac{\left(\min\left\{ r + u, \sqrt{u^2 + v^2 - r^2} + v \right\} \right)^2 - |r^2 - u^2|}{\min\left\{ r + u, \sqrt{u^2 + v^2 - r^2} + v \right\}}. \tag{60}$$

$$\cdot f(u) f(v) uv du \, dv. \tag{61}$$

Suppose now $\min\left\{ r + u, \sqrt{u^2 + v^2 - r^2} + v \right\} = r + u$. This is true if and only if $r + u \leqslant \sqrt{u^2 + v^2 - r^2} + v$. This is again equivalent to $|2r + u - v| \leqslant u + v$. Given that $2r + u - v \geqslant -(u + v)$ always (recall that $u \geqslant 0$, the condition becomes that $2r + u - v \leqslant u + v$, i.e.

that $v \geqslant r$. This means that we can write $J(f)$ as

$$J(f)(r) = \frac{2}{r}\int_0^r \int_{\sqrt{r^2-v^2}}^{+\infty} \frac{\left(\sqrt{u^2+v^2-r^2}+v\right)^2 - \left|r^2-u^2\right|}{\sqrt{u^2+v^2-r^2}+v} \cdot \tag{62}$$

$$\cdot f(u) f(v) uvdu\, dv \tag{63}$$

$$+ \frac{2}{r}\int_r^{+\infty} \int_0^{+\infty} \frac{(r+u)^2 - \left|r^2-u^2\right|}{r+u} f(u) f(v) uvdu\, dv \tag{64}$$

$$= \frac{2}{r}\int_0^r \int_{\sqrt{r^2-v^2}}^{r} \frac{\left(\sqrt{u^2+v^2-r^2}+v\right)^2 - r^2 + u^2}{\sqrt{u^2+v^2-r^2}+v} \cdot \tag{65}$$

$$\cdot f(u) f(v) uvdu\, dv \tag{66}$$

$$+ \frac{2}{r}\int_0^r \int_r^{+\infty} \frac{\left(\sqrt{u^2+v^2-r^2}+v\right)^2 - u^2 + r^2}{\sqrt{u^2+v^2-r^2}+v} \cdot \tag{67}$$

$$\cdot f(u) f(v) uvdu\, dv \tag{68}$$

$$+ \frac{2}{r}\int_r^{+\infty} \int_0^{r} \frac{(r+u)^2 - r^2 + u^2}{r+u} f(u) f(v) uvdu\, dv \tag{69}$$

$$+ \frac{2}{r}\int_r^{+\infty} \int_r^{+\infty} \frac{(r+u)^2 - u^2 + r^2}{r+u} f(u) f(v) uvdu\, dv \tag{70}$$

$$= \frac{2}{r}\int_0^r \int_{\sqrt{r^2-v^2}}^{r} 2\sqrt{u^2+v^2-r^2} f(u) f(v) uvdu\, dv \tag{71}$$

$$+ \frac{2}{r}\int_0^r \int_r^{+\infty} 2v f(u) f(v) uvdu\, dv \tag{72}$$

$$+ \frac{2}{r}\int_r^{+\infty} \int_0^{r} 2u f(u) f(v) uvdu\, dv \tag{73}$$

$$+ \frac{2}{r}\int_r^{+\infty} \int_r^{+\infty} 2r f(u) f(v) uvdu\, dv \tag{74}$$

$$= 4\int_0^{+\infty} \int_0^{+\infty} G(r,u,v) f(u) f(v) uvdu\, dv \tag{75}$$

□

Exercise 3

Finally, we consider the loss term of the collision operator, that is the part:

$$\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} ff' B(v-v_*,\omega) d\omega dv_*$$

in the right-hand side of the Boltzmann equation. In the case of hard sphere interactions with a solution which is homogeneous and radial in velocity, we saw that this term can be written as:

$$\int_0^{+\infty}\int_0^{\pi}\int_0^{\pi} f(t,r)f(t,r_1)V(r,r_1,\theta,\theta_1)r_1^2 d\theta d\theta_1 dr_1 = f(t,r)L(f)(t,r),$$

with

$$L(f)(t,r) = \int_0^{+\infty} P(r,r_1)f(t,r_1)r_1^2 dr_1,$$

and

$$P(r,r_1) = \int_0^\pi \int_0^\pi \Big|r_1 \cos\theta_1 - r\cos\theta\Big| \sin\theta \sin\theta_1 d\theta d\theta_1. \tag{76}$$

Show that the quantity P in (76) can be expressed as:

$$P(r,r_1) = \big(2r + \frac{2r_1^2}{3r}\big)\mathbb{1}_{r_1 \leqslant r} + \big(2r_1 + \frac{2r^2}{3r_1}\big)\mathbb{1}_{r_1 > r}.$$

Proof. Let P be defined as in (76). From the definition, it is clear that P is symmetric in r and r_1. We then first assume that $r \geqslant r_1$ and then obtain the formula by symmetry. We first perform the change of variables

$$\begin{cases} x = \cos\theta, \\ y = \cos\theta_1. \end{cases} \tag{77}$$

The Jacobian is given by $dxdy = \sin\theta \sin\theta_1 d\theta d\theta_1$; as a consequence we get

$$\begin{aligned}
P(r,r_1) &= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} |r_1 \cos\theta_1 - r\cos\theta| \sin\theta \sin\theta_1 d\theta\ d\theta_1 && (78)\\
&\quad + \int_0^{\frac{\pi}{2}} \int_{\frac{\pi}{2}}^{\pi} |r_1 \cos\theta_1 - r\cos\theta| \sin\theta \sin\theta_1 d\theta\ d\theta_1 && (79)\\
&\quad + \int_{\frac{\pi}{2}}^{\pi} \int_0^{\frac{\pi}{2}} |r_1 \cos\theta_1 - r\cos\theta| \sin\theta \sin\theta_1 d\theta\ d\theta_1 && (80)\\
&\quad + \int_{\frac{\pi}{2}}^{\pi} \int_{\frac{\pi}{2}}^{\pi} |r_1 \cos\theta_1 - r\cos\theta| \sin\theta \sin\theta_1 d\theta\ d\theta_1 && (81)\\
&= \int_0^1 \int_0^1 |r_1 y - rx|\, dx\ dy + \int_0^1 \int_{-1}^0 |r_1 y - rx|\, dx\ dy && (82)\\
&\quad + \int_{-1}^0 \int_0^1 |r_1 y - rx|\, dx\ dy + \int_{-1}^0 \int_{-1}^0 |r_1 y - rx|\, dx\ dy && (83)\\
&= 2\int_0^1 \int_0^1 (|rx - r_1 y| + |rx + r_1 y|)\, dx\ dy && (84)\\
&= \frac{2}{rr_1} \int_0^{r_1} \int_0^r (|X - Y| + |X + Y|)\, dX\ dY && (85)\\
&= \frac{2}{rr_1} \int_0^{r_1} \left(\int_0^Y 2Y dX + \int_Y^r 2X dX \right) dY && (86)\\
&= \frac{2}{rr_1} \int_0^{r_1} \left(Y^2 + r^2\right) dY = 2\left(r + \frac{r_1^2}{3r}\right). && (87)
\end{aligned}$$

This gives us the result.

□